Last Time: $L: V \to W$ linear

$\ker(L) = \{v \in V : L(v) = 0_W\}$.

$\operatorname{ran}(L) = \{L(v) : v \in W\}$.

Prop: $L: V \to W$ linear.

① L is injective iff $\ker(L) = 0$

② L is surjective iff $\operatorname{ran}(L) = W$.

NB: A bijective linear map (i.e. a linear map which is both injective and surjective) is a linear isomorphism... Very important...

Prop (Rank-Nullity Formula): Suppose $L: V \to W$ is a linear map. Then we have

$$\dim(V) = \underbrace{\dim(\ker(L))}_{\text{nullity}(L)} + \underbrace{\dim(\operatorname{ran}(L))}_{\text{rank}(L)}.$$

Pf: Let $L: V \to W$ be a linear map. Let B_0 be a basis for $\ker(L) \leq V$. Now B_0 extends to a basis $B \supseteq B_0$ for V. Let $A := B \setminus B_0$.

Claim: $L(A) := \{L(a) : a \in A\} \subseteq \operatorname{ran}(L)$ is a basis of $\operatorname{ran}(L)$.

Note $L(A)$ spans $\operatorname{ran}(L)$ (because every element of $\operatorname{ran}(L)$ can be expressed as:

$$\left[L\left(\sum_{b \in B} c_b b\right)\right] = L\left(\sum_{b \in B_0} c_b b + \sum_{a \in A} c_a a\right)$$

$$= L\left(\sum_{b \in B_0} c_b b\right) + L\left(\sum_{a \in A} c_a a\right)$$

notation trick...

Point: Break up

the sum by inclusion in B_0 or A

$$= \sum_{b \in B_0} c_b L(b) + \sum_{a \in A} c_a L(a)$$

$$= 0_W + \sum_{a \in A} c_a L(a)$$

$$= \sum_{a \in A} c_a L(a)$$

b/c every elt of V can be expressed in this way (i.e. using basis B)

So $L(A)$ spans $\mathrm{ran}(L)$. To see $L(A)$ is linearly indep., suppose $\sum_{i=1}^{n} c_i L(a_i) = 0_W$.

Thus $L\left(\sum_{i=1}^{n} c_i a_i\right) = 0_W$, so $\sum_{i=1}^{n} c_i a_i \in \ker(L)$.

Hence $\sum_{i=1}^{n} c_i a_i + \boxed{\sum_{b \in B_0} 0 b}$ is the unique expression for $\sum_{i=1}^{n} c_i a_i$ in terms of the basis B.

But $\sum_{i=1}^{n} c_i a_i \in \ker(L)$, so $c_i = 0$ for all i.

Hence $L(A)$ is linearly independent. Thus $L(A)$ is a basis for $\mathrm{ran}(L)$. But

$$B_0 \cup A = B, \text{ so } \# B = \# B_0 + \# A.$$

On the other hand, $\# B = \dim(V)$, $\# B_0 = \dim(\ker(L))$, $\# L(A) = \dim(\mathrm{ran}(L))$. Hence, we have

$\boxed{\dim(V) = \dim(\ker(L)) + \# A}$ Now we must show $\# A = \# L(A)$. If $\# A > \# L(A)$, then there are $a, a' \in A$ with $L(a) = L(a')$; But then $L(a - a') = 0_W$

So $a - a' \in \ker(L)$, so $B_0 \cup \{a, a'\}$ is linearly dependent, contradicting our assumption $B = B_0 \cup A \supseteq B_0 \cup \{a, a'\}$ is a basis... Thus $\# A \leq \# L(A) \leq \# A$.

Hence
$$\begin{aligned}\dim(V) &= \dim(\ker(L)) + \# A \\ &= \dim(\ker(L)) + \# L(A) \\ &= \dim(\ker(L)) + \dim(\operatorname{ran}(L)) \\ &= \text{nullity}(L) + \text{rank}(L).\end{aligned}$$ $\boxtimes$

Ex: Suppose $L : V \to \mathbb{R}^{15}$ has $\text{nullity}(L) = 7$ and L is surjective. Q: what is $\dim(V)$?

Sol: by the rank-nullity formula, $\dim(V) = \text{nullity}(L) + \text{rank}(L)$. $\text{nullity}(L) = 7$, and $\operatorname{ran}(L) = \mathbb{R}^{15}$, so $\text{rank}(L) = 15$.

Hence $\dim(V) = 7 + 15 = 22$. $\boxtimes$

Ex: Suppose $L : \mathbb{R}^3 \to \mathbb{R}^2$ is linear.

Q: what can $\text{rank}(L)$ and $\text{nullity}(L)$ be?

Sol: The rank-nullity formula yields

$$\left[3 = \dim(\mathbb{R}^3) = \text{nullity}(L) + \text{rank}(L)\right] \rightsquigarrow 0 \leq \text{rank}(L) \leq 2$$

OTOH, $\text{rank}(L) \in \{0, 1, 2\}$. $\rightsquigarrow$

If $\text{rank}(L) = 1$: $\text{nullity}(L) = 3 - 1 = 2$

If $\text{rank}(L) = 2$: $\text{nullity}(L) = 3 - 2 = 1$

If $\text{rank}(L) = 0$: $\text{nullity}(L) = 3 - 0 = 3$

Thus $[1 \leq \text{nullity}(L) \leq 3]$. $\boxtimes$

Point: Every linear transformation from $\mathbb{R}^3 \to \mathbb{R}^2$ has a nontrivial kernel!

In fact ...

Cor: If $m < n$ and $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then L is not injective.

Pf: $\dim(\mathrm{dom}(L)) = \dim(\ker(L)) + \dim(\mathrm{ran}(L))$, so
$n = \dim(\ker(L)) + \dim(\mathrm{ran}(L))$. Moreover,
$0 \le \dim(\mathrm{ran}(L)) \le \dim(\mathbb{R}^m) = m$ (b/c $\mathrm{ran}(L) \le \mathbb{R}^m$).
Hence $n = \dim(\ker(L)) + \dim(\mathrm{ran}(L)) \le \dim(\ker(L)) + m$.
So $0 < n - m \le \dim(\ker(L))$. Hence $\ker(L) \neq \{0_V\}$,
so L is not injective. $\boxtimes$

Ex: Let $L : V \to W$ be a linear map. Define for all $U \le W$, $L^{-1}U := \{v \in V : L(v) \in U\}$. Prove $L^{-1}U \le V$. **Q:** What can you say about $\dim(L^{-1}U)$?

Hint: Rank nullity formula, apply to $L : L^{-1}U \to U$ ...

Lem: Suppose $L : V \to W$ and $Q : W \to U$ are linear.
Then $Q \circ L : V \to U$ is linear.
(i.e. compositions of linear maps are linear maps).

Recall: The composition of two functions $f : A \to B$ and $g : B \to C$ is the map $g \circ f : A \to C$ defined by $(g \circ f)(x) = g(f(x))$ for all $x \in A$.

Reminder: Composition of functions is associative...
i.e. $h \circ (g \circ f) = (h \circ g) \circ f$.

Pf (Lem): Exercise :) $\boxtimes$

Point: Compositions of linear maps can be used to produce more linear maps :)

Defn: A *linear isomorphism* of vector spaces V and W is a linear map $L: V \to W$ which is bijective.

V and W are *isomorphic* when there is an isomorphism between them (and we write $V \cong W$).

Ex: Claim $\mathbb{R}^4 \cong \mathrm{Mat}_{2\times 2}(\mathbb{R})$.

pf: We construct an explicit isomorphism.

Look at bases $\mathcal{E}_4 = \{e_1, e_2, e_3, e_4\}$ and

$$B = \left\{ b_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, b_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, b_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, b_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Left to you: B is a basis of $\mathrm{Mat}_{2\times 2}(\mathbb{R})$.

[Define $L: \mathbb{R}^4 \to \mathrm{Mat}_{2\times 2}(\mathbb{R})$ by linearly extending $L(e_i) = b_i$ for $1 \le i \le 4$.] Left to you:

$$L\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

To see L is injective:

$$* \; L\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}^{*} \iff \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \iff x = y = z = w = 0$$

$$\iff \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$ Hence $\ker(L) = 0$.

To see L is surjective, note $\mathrm{ran}(L) \supseteq B$, which is a basis for $\mathrm{Mat}_{2\times 2}(\mathbb{R})$, so $\mathrm{ran}(L) = \mathrm{Mat}_{2\times 2}(\mathbb{R})$ yields L is surjective.

Hence L is bijective and linear, so L is an isomorphism, yielding $\mathbb{R}^4 \cong \mathrm{Mat}_{2\times 2}(\mathbb{R})$. $\blacksquare$

NB: Nothing special about this example...
All we needed to make this argument was that the vector spaces had the same dimension!

Prop: Two vector spaces are isomorphic if and only if they have the same dimension.

pf: Let V and W be vector spaces.

($\Rightarrow$): Assume V and W are isomorphic. Thus there is an isomorphism $L: V \to W$. Let $\mathcal{B}$ be a basis of V. $L(\mathcal{B})$ is a basis for W by the same argument we made when proving the rank-nullity formula: $\mathcal{B} = \emptyset \cup \mathcal{B}$ and $\emptyset$ is a basis for $\{0_V\} = \ker(L)$. Hence, by injectivity $\dim(V) = \#\mathcal{B} = \#L(\mathcal{B}) = \dim(W)$.

($\Leftarrow$): Assume V and W have the same dimension. Let $\mathcal{B}$ be a basis of V and A a basis of W. By assumption, $\#\mathcal{B} = \dim(V) = \dim(W) = \#A$. Let f be any bijection $f: \mathcal{B} \to A$. Extend f linearly to $F: V \to W$ (by a previous proposition). Because A is a basis (hence linearly independent), one can show $\ker(F) = 0$ (i.e. F is injective). OTOH $\operatorname{ran}(F) \supseteq F(\mathcal{B}) = A$ so $\operatorname{ran}(F) = W$. Hence F is bijective. $\blacksquare$